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EXTREME BOUNDARIES OF CONVEX BODIES IN l_2

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ABSTRACT

Every uncountable complete separable metric space is homeomorphic to the set of extreme points (in the weak topology) of a bounded closed convex body in l_2.

1. J. Lindenstrauss and R. R. Phelps proved in [7, Corollary 1.2] that the set of extreme points of a bounded closed convex body C in an infinite dimensional reflexive space E is uncountable. It has been previously known that if E is separable then ext C with the weak topology is separable and metrizable with a complete metric. It has been conjectured by the above authors that, conversely, for any uncountable complete separable metric space X there is a bounded closed convex body in l_2 whose set of extreme points in the weak topology is homeomorphic to X. This conjecture was established by them with the additional assumption that X is compact (see [7, Proposition 1.3], where a somewhat weaker result is proved). We are now able to prove their conjecture by relying on some ideas from the proof of Proposition 1.3 in [7] and on Haydon's proof [5] of a result of Choquet [2, Theorem 29.9], which states that every complete separable metric space is homeomorphic to the set of extreme points of some simplex. We wish to express our thanks to Professor J. Lindenstrauss for suggesting to us the problem and for communicating to us the proof he and R. R. Phelps had for the case of a compact metric space.

We use the notation of [1] with a few exceptions which are noted below. All the linear spaces considered are real. The set of extreme points of a convex set C is denoted ext C. If A is a subset of the topological space T, then $\mathrm{cl}_T A$ denotes the closure of A in T. For a compact Hausdorff space T, $P(T)$ represents the set of probability Radon measures on T with the w^*-topology it

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inherits from $C(T)^* = M(T)$ — the space of all Radon measures on T. For $t \in T$, $\delta(t)$ is the Dirac measure at t. If X is a Banach space, we denote by $B(X)$ its closed unit ball.

2. We begin with a technical result on compactifications.

LEMMA 1. *Let X be a separable metric space and $\{X_i\}_{i=1}^4$ closed subsets of X such that*

$$\bigcup_{i=1}^{4} X_i = X,\ X_1 \cap X_3 = X_1 \cap X_4 = X_2 \cap X_4 = \varnothing.$$

Then X can be topologically embedded as a dense subset of a compact metric space T such that

$$\mathrm{cl}_T(X_i \cap X_{i+1}) = \mathrm{cl}_T X_i \cap \mathrm{cl}_T X_{i+1}$$

for $1 \leqq i \leqq 3$ *and*

$$\mathrm{cl}_T X_1 \cap \mathrm{cl}_T X_3 = \mathrm{cl}_T X_1 \cap \mathrm{cl}_T X_4 = \mathrm{cl}_T X_2 \cap \mathrm{cl}_T X_4 = \varnothing.$$

PROOF. Let ρ be a homeomorphism of X into the Hilbert cube and f be a continuous function from X to $[0, 1]$ such that $f|X_1 \cap X_2 = 0$, $f|X_2 \cap X_3 = \frac{1}{2}$ and $f|X_3 \cap X_4 = 1$. Then $\Psi(X) = (f(X), \rho(X))$ defines a homeomorphism of X into the Hilbert cube such that the closures in $[0, 1]^{\aleph_0}$ of $X_1 \cap X_2$, $X_2 \cap X_3$ and $X_3 \cap X_4$ are mutually disjoint. Thus we may suppose that X is embedded into a compact metric space Y such that

$$\begin{aligned}\mathrm{cl}_Y(X_1 \cap X_2) \cap \mathrm{cl}_Y(X_2 \cap X_3) = \mathrm{cl}_Y(X_1 \cap X_2) \cap \mathrm{cl}_Y(X_3 \cap X_4)\\ = \mathrm{cl}_Y(X_2 \cap X_3) \cap \mathrm{cl}_Y(X_3 \cap X_4) = \varnothing.\end{aligned}$$

Now, let

$$Y_1 = X_1 \cup \mathrm{cl}_Y(X_1 \cap X_2),$$

$$Y_2 = X_2 \cup \mathrm{cl}_Y(X_1 \cap X_2) \cup \mathrm{cl}_Y(X_2 \cap X_3),$$

$$Y_3 = X_3 \cup \mathrm{cl}_Y(X_2 \cap X_3) \cup \mathrm{cl}_Y(X_3 \cap X_4),$$

$$Y_4 = X_4 \cup \mathrm{cl}_Y(X_3 \cap X_4)$$

and let Z_i, $1 \leqq i \leqq 4$, be a compact metric space which contains Y_i as a dense subset. Let Z be the disjoint union of Z_1, Z_2, Z_3 and Z_4. Identify in Z the corresponding points of $\mathrm{cl}_Y X_i \cap X_{i+1}$ (in Z_i and Z_{i+1}), $1 \leqq i \leqq 3$; one gets a compact metrizable space T. It is readily checked that T satisfies the requirements.

The next lemma is an ad-hoc modification of the Choquet-Haydon theorem [2, p. 183] mentioned in the introduction. The proof differs from that of [5] only by some technical details.

LEMMA 2. *Let X be a complete separable metric space and $\{X_i\}_{i=1}^4$ closed subsets of X such that*

$$\bigcup_{i=1}^{4} X_i = X,\ X_1 \cap X_3 = X_1 \cap X_4 = X_2 \cap X_4 = \varnothing.$$

Then there exists a metrizable simplex K, a homeomorphism ϕ of X onto ext K *and closed faces $\{F_i\}_{i=1}^4$ of K such that $\phi(K_i) =$* ext F_i, $1 \leqq i \leqq 4$.

PROOF. Let T be as in Lemma 1. By [6, p. 430] there is a decreasing sequence $\{G_n\}_{n=0}^{\infty}$ of open subsets of T such that $G_0 = T$ and $\bigcap_{n=0}^{\infty} G_n = X$. Let $\{\varepsilon_r\}_{r=1}^{\infty}$ be a sequence from (0, 1) which converges to zero. Each G_n is the union of a sequence $\{H_n^r\}_{r=1}^{\infty}$ of compact sets. For each $x \in H_n^r$ choose a neighbourhood of diameter $< 2^{-n}\varepsilon_r$ whose closure is contained in G_n and which is included in $T \setminus \bigcup_{j \neq i} \mathrm{cl}_T X_j$ if $x \in T \setminus \bigcup_{j \neq i} \mathrm{cl}_T X_j$ or is included in $T \setminus \bigcup_{i \neq i, j \neq i+1} \mathrm{cl}_T X_j$ if $x \in \mathrm{cl}_T X_i \bigcap \mathrm{cl}_T X_{i+1}$. By passing to finite subcovers of each H_n^r we get a null-sequence $\{\eta_k^n\}_{k=0}^{\infty} \subset (0,1)$ and open sets $\{G_k^n\}_{k=0}^{\infty}$ such that $G_n = \bigcup_{k=0}^{\infty} G_k^n$, diam $(G_k^n) < 2^{-n}\eta_k^n$, $\mathrm{cl}_T G_k^n \subset G_n$ and each of the sets G_k^n either ntersects only one of the sets $\mathrm{cl}_T X_i$ or, if it meets two of them, say $\mathrm{cl}_T X_i$ and $\mathrm{cl}_T X_{i+1}$, then

$$G_k^n \cap \mathrm{cl}_T X_i \cap \mathrm{cl}_T X_{i+1} \neq \varnothing.$$

No G_k^n meets more than two of the sets $\mathrm{cl}_T X_i$. For each n and k let $g_k^n \in C(T)$ satisfy: $0 \leqq g_k^n \leqq 2^{-k}$, g_k^n vanishes identically on $T \setminus G_k^n$ and is strictly positive on G_k^n. Define $h_k^n \in C(T)$ as follows: $h_k^n(x) = 0$ if $x \in T \setminus G_n$,

$$h_k^n(x) = g_k^n(x)\Big(\sum_{k=0}^{\infty} g_k^n(x)\Big)^{-1} \text{ if } x \in G_n.$$

Then $\Sigma_{k=0}^{\infty} h_k^n = \chi_{G_n}$, diam (supp $h_k^n) \leqq 2^{-n}\eta_k^n$, and h_k^n is not identically zero on at most two of the sets $\mathrm{cl}_T X_i$. If h_k^n is not trivial on $\mathrm{cl}_T X_i$ and $\mathrm{cl}_T X_{i+1}$, then it is not trivial on $\mathrm{cl}_T X_i \cap \mathrm{cl}_T X_{i+1}$. Define $p_k^n = h_k^n \chi_{T \setminus G_{n+1}}$. If $p_k^n \neq 0$ on $\mathrm{cl}_T X_i \cap \mathrm{cl}_T X_{i+1}$, choose points $x_k^n, y_k^n \in X_i \cap X_{i+1} \cap$ supp h_k^n, $x_k^n \neq y_k^n$ and if $p_k^n \neq 0$ on $\mathrm{cl}_T X_i$ only, then choose $x_k^n, y_k^n \in X_i \cap$ supp h_k^n. If p_k^n is identically zero, let x_k^n, y_k^n be any two points of X. For $\mu \in M(T)$ denote

$$\gamma(\mu) = \mu - \frac{1}{2}\sum_{k,n=0}^{\infty} \int p_k^n d\mu \ \ (\delta(x_k^n) + \delta(y_k^n))$$

and let $E = \{\gamma(\mu): \mu \in M(T), |\mu|(X) = 0\}$. Then E is a w^*-closed subspace of $M(T)$ by [5]. Let ω be the restriction to $P(T)$ of the quotient map from $M(T)$ onto $M(T)/E$ and $K = \omega(P(T))$. Then one sees as in [5] that K is a simplex and $\phi = \omega \circ \delta$ is a homeomorphism of X onto ext K. We have to show that the closed convex hull of $\phi(X_i)$ is a face of K. To this end it suffices to show that for each point in $\mathrm{cl}_K \phi(X_i)$, the support of the unique maximal measure which represents it is contained in $\mathrm{cl}_K \phi(X_i)$(cf. [3, Theorem 3.3]). Clearly $\mathrm{cl}_K \phi(X_i) = \phi(\mathrm{cl}_T X_i)$, so let $t \in \mathrm{cl}_T X_i \setminus X_i$ and suppose $t \in G_n \setminus G_{n+1}$. Then

$$\gamma(t) = \delta(t) - \frac{1}{2}\sum_{k=0}^{\infty} p_k^n(t)(\delta(x_k^n) + \delta(y_k^n)),$$

thus

$$\phi(t) = \frac{1}{2}\sum_{k=0}^{\infty} P_k^n(t) \ (\phi(x_k^n) + \phi(y_k^n)). \tag{1}$$

Since x_k^n, $y_k^n \in X_i$ if $p_k^n(t) \neq 0$, (1) gives the maximal measure representing $\phi(t)$ and it is plain that its support is contained in $\mathrm{cl}_K \phi(X_i)$. This concludes the proof of the lemma.

For the remainder of the paper we shall use the following notation: $I = [0, 1]$, $I_1 = [0, 2^{-2}]$, $I_2 = [2^{-2}, 2^{-1}]$, $I_3 = [2^{-1}, 3 \cdot 2^{-2}]$ and $I_4 = [3 \cdot 2^{-2}, 1]$.

Lemma 3. *Let X be an uncountable complete separable metric space. Then there exist a metrizable simplex K whose set of extreme points is homeomorphic to X and a continuous affine map g of K onto $P(I)$ such that $g(\mathrm{ext}\, K) \subset \bigcup_{j=1}^4 P(I_j)$.*

Proof. By [6, p. 444] X contains a subset C homeomorphic to the Cantor set. Let C_1, C_2 be two disjoint subsets of C homeomorphic to C. By normality there are disjoint open subsets G_1, G_2 of X which contain C_1, C_2 respectively. Denote $H_i = X \setminus G_i$. Let Y_i be an uncountable open subset of G_i such that $\mathrm{cl}_X Y_i \subset G_i$ and $G_i \setminus \mathrm{cl}_X Y_i$ is uncountable. Now let $X_1 = \mathrm{cl}_X Y_1$, $X_2 = H_2 \setminus Y_1$, $X_3 = H_1 \setminus Y_2$ and $X_4 = \mathrm{cl}_X Y_2$. Clearly $X = \bigcup_{i=1}^4 X_i$, $X_1 \cap X_3 = X_1 \cap X_4 = X_2 \cap X_4 = \emptyset$, $\{X_i\}_{i=1}^4$ are closed subsets of X and $X \setminus \bigcup_{j \neq i} X_j$ is uncountable, $1 \leq i \leq 4$. Consider now the simplex K given for X and $\{X_i\}_{i=1}^4$ by Lemma 2. We shall identify X with ext K. Denote by F_i the closed convex hull of X_i. Let A be a subset of $X \setminus \bigcup_{i=2}^4 X_i$ homeomorphic to the Cantor set (see [6, p. 408, p. 444]) and g_1 a continuous function from A onto I_1. By [1, p. 108] and [4, p. 32, p. 78], g_1 admits a continuous affine extension, which will be denoted by g_1 too, from F_1 onto $P(I_1)$ such that $g_1(F_1 \cap F_2) = \{\delta(2^{-2})\}$. In a similar manner one constructs continuous affine functions g_k from F_k onto $P(I_k)$, $2 \leqq k \leqq 4$, such that $g_2(F_1 \cap F_2) = \{\delta(2^{-2})\}$, $g_2(F_2 \cap F_3) = g_3(F_2 \cap F_3) = \{\delta(2^{-1})\}$ and $g_3(F_3 \cap F_4) = g_4(F_3 \cap F_4) = \{\delta(3 \cdot 2^{-2})\}$. The simplexes $P(I_k)$ can be considered as closed faces of $P(I)$. One more use of [4, p. 32] yields the desired function g.

3. We may prove now our main result.

THEOREM. *Every uncountable complete separable metric space is homeomorphic to the set of extreme points (with the weak topology) of a bounded closed convex body in l_2.*

PROOF. Let $H_1 = H_2 = l_2$ and $H = H_1 \oplus H_2$. By the Hahn-Mazurkiewicz theorem, there is a (weakly) continuous map ψ' of I into H such that $\psi'(I_1) = B(H_1) \times \{0\}$, $\psi'(I_2 \cup I_3) = \{(0, 0)\}$ and $\psi'(I_4) = \{0\} \times B(H_2)$. Denote its unique continuous affine extension from $P(I)$ to H by ψ', too. Consider now the simplex K and the map $g = K \to P(I)$ given by Lemma 3 with $X = \operatorname{ext} K$. The composition $\psi = \psi' \circ g$ maps the closed faces $F_j = g^{-1}(P(I_j))$ as follows: $\psi(F_1) = B(H_1) \times \{0\}$, $\psi(F_2 \cup F_3) = \{(0, 0)\}$, $\psi(F_4) = \{0\} \times B(H_2)$. Define the following functions on I:

$$\alpha(t) = \begin{cases} 0 \quad , & 0 \leqq t \leqq 2^{-2}, \\ 4t - 1, & 2^{-2} \leqq t \leqq 2^{-1}, \\ 1 \quad , & 2^{-1} \leqq t \leqq 1, \end{cases}$$

$$\beta(t) = \begin{cases} 1 \quad , & 0 \leqq t \leqq 2^{-1}, \\ 3 - 4t, & 2^{-1} \leqq t \leqq 3 \cdot 2^{-2}, \\ 0 \quad , & 3 \cdot 2^{-2} \leqq t \leqq 1. \end{cases}$$

We shall use α and β to denote also their unique extensions as continuous affine functions on $P(I)$.

Put $K_1 = \operatorname{conv}(F_1 \cup F_2)$, $K_2 = \operatorname{conv}(F_3 \cup F_4)$. Choose two sequences $\{f_n^1\}_{n=1}^{\infty}$, $\{f_n^2\}_{n=1}^{\infty}$ in the closed unit ball of $A(K)$ such that $f_n^1 | F_1 = 0$, $f_n^2 | F_4 = 0$ for $n = 1, 2, \cdots$, and $\{f_n^1 | K_2\}_{n=1}^{\infty}$, $\{f_n^2 | K_1\}_{n=1}^{\infty}$ are dense in the closed unit ball of $A(K_2)$, $A(K_1)$, respectively. Define now $\phi \colon K \to H$ as follows:

$$\phi(x) = \{(2\alpha(g(x)), 8^{-1}f_1^1(x), \cdots, 8^{-n}f_n^1(x), \cdots),$$
$$(2\beta(g(x)), 8^{-1}f_1^2(x), \cdots, 8^{-n}f_n^2(x), \cdots)\},$$

and let $h: K \to H$ be given by $h(x) = \phi(x) + \psi(x)$. We claim that $C = h(x)$ is a bounded closed convex body in H and that ext C is homeomorphic to $X = \text{ext } K$.

First we are going to show that the restriction of h to $K_1 \cup K_2$ is a homeomorphism. Since h is clearly continuous, we have only to show that h is one-to-one on $K_1 \cup K_2$. If $x, y \in K_1$, the $(n+1)$th H_2-coordinates of $h(x)$ and $h(y)$ are $8^{-n}f_n^2(x)$ and $8^{-n}f_n^2(y)$, repectively; thus $h(x) = h(y)$ if and only if $x = y$. Similarly one sees that $h|K_2$ is one-to-one. If $x \in K_1$ and $y \in K_2$, then the first H_2-coordinate of $h(x)$ is 2, while the same coordinate of $h(y)$ is 2 only if $g(y) = \delta(2^{-1})$, and in this case $y \in F_2 \cap F_3 \subset K_1$.

We shall now show that $h(X) = \text{ext } C$. Obviously ext $C \subset h(X)$. Let $x \in X$ and suppose $x \in K_1$. Assume

$$h(x) = \lambda h(y) + (1-\lambda)h(z) \tag{*}$$

with $0 \leqq \lambda \leqq 1$, $y, z \in K$. Since $K = \text{conv}\,(K_1 \cup K_2)$, by applying the same argument as at the end of the last paragraph, we get $y, z \in K_1$. Now (*) yields $f_n^2(x) = \lambda f_n^2(y) + (1-\lambda) f_n^2(z)$, $n = 1, 2, \cdots$. By the choice of $\{f_n^2\}_{n=1}^{\infty}$, $x = \lambda y + (1-\lambda)z$. Thus $x = y = z$ and $h(x) \in \text{ext } C$. A similar argument works in the case $x \in K_2$, hence $h(X) \subset \text{ext } C$.

It remains to show that int $C \neq \emptyset$. One shows that $\{(1,0,0,\cdots),(1,0,0,\cdots)\} \in$ int C as in [7, Proposition 1.3].

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